## Three-Dimensional Method of Characteristics – A Survey

Practical use of the three-dimensional method of characteristics has been limited due to basic differences from the two-dimensional method making the extension to a third dimension difficult. In the two-dimensional case the governing partial differential equations when written in the characteristic form become ordinary differential equations which hold along the characteristics. As the equations essentially drop one order of magnitude in this method, in the case of three independent variables the original equations remain partial differential equations which though containing derivatives in the characteristic surfaces only, still contain derivatives along two directions in the surface.

For steady three-dimensional supersonic flow there exist two families of characteristic surfaces: one such family consists of the surfaces of streamlines through each point, and the other of wave surfaces whose envelope forms a conoid which determines the regions of dependence and of influence of the point. This Mach conoid is represented locally as a Mach cone. The lines along which the characteristic surfaces touch the Mach conoid are called bicharacteristics. Whereas the calculation scheme is more-or-less uniquely determined in the two-dimensional case as two characteristic lines pass through each point, in the three-dimensional case we have an infinite number of bicharacteristics forming the Mach conoid; so various types of computational schemes can be attempted.

Holt (1) presents three three-dimensional methods. Both the Godunov and BVLR schemes have their origins in the method of characteristics. Holt notes that "in principle all nonstationary problems in gas dynamics could be solved by a method of characteristics; but the method does not lend itself easily to machine computations."

The Godunov (2) scheme involves a breakdown of the computation space into cells

across the boundaries of which pressure and velocity are in general discontinuous. For three-dimensional flow, the flow field is divided into strips parallel to the axis of symmetry which are treated as channels. The problem of treating a contact discontinuity is solved between cell boundaries along each strip and between adjoining strips.

The BVLR method (3) is a finite difference method in which one variable becomes the time-like marching variable along which the solution is solved for in parallel planes using a double-sweep elimination technique. The relations along characteristic lines by which the solution advances are replaced by equivalent conditions along lines running in coordinate directions.

A bicharacteristic method by Butler (4) and discussed by Holt makes use of the condition that an infinite number of bicharacteristics pass through any given point by using difference relations along four bicharacteristic directions to result in two independent relations involving only derivatives in the bicharacteristic directions at that point. To do this, one additional relation is needed, and the relations along an ordinary curve through the point and lying within the region of dependence are used. On this curve also the derivatives in directions other than that of the curve are eliminated by a combination of the bicharacteristic relations.

Vadyak and Hoffman (5) have applied Butler's method to study the flowfield in propulsive nozzles based more directly on the second-order bicharacteristic algorithm developed by Ransom, Hoffman and Thompson (6). An inverse marching scheme is used, the solution being

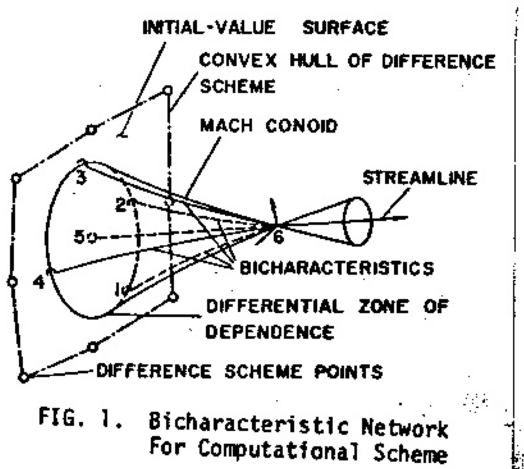


FIG. 1. Bicharacteristic Network For Computational Scheme

obtained on a series of space-like surfaces. Flow computations are performed in a local coordinate system aligned such that its x-axis coincides with the streamline through the point (i.e., marching is in the local x-direction). The position and velocity vectors are transformed into the local coordinates before the unit process computations are begun.

First the location of the solution point is determined by extending the streamline from the mesh point in the initial-value surface by simple finite differencing ($dx = u\,dt$). Then four bicharacteristics are extended from the solution point back to the initial-value surface. The initial-value surface is represented by a polynomial function the coefficients of which having been determined by a least squares fit. The intersections of the bicharacteristics with the initial-value surface are found by an iterative procedure using the finite-difference approximation of Butler's parametric form of representing a bicharacteristic. The flow properties at these intersections are found by interpolation using polynomial representations for these properties in the initial-value plane.

Next the five compatibility equations are solved to find the five flow properties at the solution point. Two of the compatibility equations are applicable on the stream surface; the other three are linear combinations of the compatibility relations for the wave surfaces and the relation along the streamline eliminating cross-derivative terms (such as $\frac{\partial u}{\partial n}$, i.e., terms involving differentiation in the wave surface normal to the bicharacteristic direction). The five equations are solved using a Gaussian elimination technique in a predictor-corrector method.

The size of the marching step is determined by the CFL stability criterion (for linear hyperbolic equations) applied at every streamline point with the actual step being taken as that at the most restrictive point. The use of the same

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marching step produces a smooth solution surface.

Holt notes as disadvantages of Butler's scheme that:

① at each new point in the network, a new coordinate system must be determined that is not derived from the previous calculations nor related in a definite way to the fixed coordinate system, making the geometry complicated.

② according to a previous researcher (Jitt, 1959) who studied existence and uniqueness properties of hyperbolic equations, only two bicharacteristic directions (and their compatibility conditions) are independent. This brings into question the validity of calculations along more than two bicharacteristic directions.

Chushkin (7) notes the disadvantages of Butler's method as being

① the great number of interpolations necessary for determining flow properties at the four intersection points of the bicharacteristics with the initial-value surface, with the accompanying large amount of computer time and accumulated numerical error; and

② that the calculation of the boundary points becomes too cumbersome.

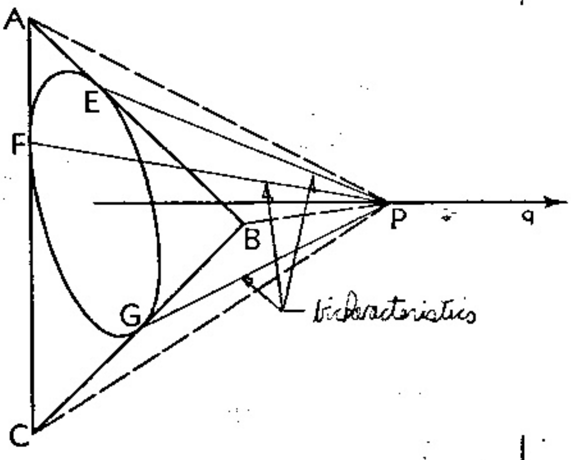
Nevertheless, Vadyak and Hoffman use this bicharacteristic method in the study mentioned and give the computation time required for a network of 9-radial and 15-circumferential stations marching through 81 solution surfaces of 25 minutes of CPU time and for a solution through 20 solution surfaces of 9 minutes of CPU time on a CDC-6500/6600 system.

Holt further discusses 2 other methods of characteristics. An optimal method based on two bicharacteristic families and one non-characteristic family proposed by Bruhn and Haack (1958) and later developed by Schaetz (1961) uses for directions two bicharacteristic directions in the plane of the velocity vector and the principal normal

of the streamline, the velocity direction, and the binormal to the streamline. These directions change from point to point, but three of the directions are coplanar and the fourth normal to the plane. Schaetz has applied this method to unsteady nozzle flow.

The last method mentioned by Holt is the "near characteristic" method of Sauer (1962). This method is similar to that of Bruhn and Haack in that characteristic conditions are applied in two directions and one of the coordinate directions is taken to be non-characteristic. Sauer uses characteristic directions other than bicharacteristic in an optimal scheme based upon a rigorous derivation concerned with uniqueness and equivalence. Basically in this scheme a three-dimensional problem is treated as a sequence of two-dimensional problems. Though the three-dimensional application is only generally presented in the text, Holt feels Sauer's method is the best and most efficient means of treating steady supersonic three-dimensional problems.

Chushkin presents two tetrahedral schemes and a scheme similar to Butler's in his review. In the first cases the base of the tetrahedron lies in the initial-value surface, and the three faces are characteristic surfaces tangential to the Mach cone. The three compatibility relations on the bicharacteristics and relations for the streamline are used to calculate the flow parameters at the new point. Based on Holt's comments on the independence of only two bicharacteristic directions, these methods become as suspect at that of Butler. Chushkin

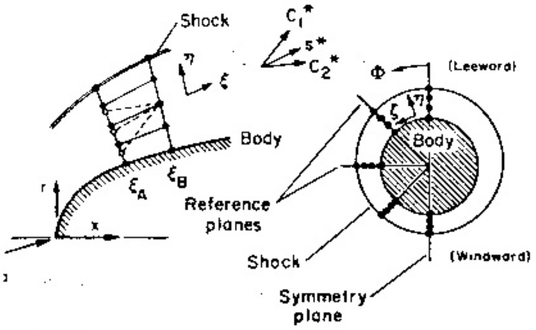


also presents a semi-characteristic scheme developed by Lin and Katskova. Other semi-characteristic or reference-plane schemes have been devised or used by Ferri (8,9), Moretti (10), and Rakich (11). Ferri (8) in his work treating the potential equation determines the intersections of the characteristic surfaces with planes parallel to the x-axis and with meridian planes, and then proceeds to calculate the variation of the velocity along the intersection lines. This method is presented in different form by Ferri (9), now solving Euler's equations in Cartesian rather cylindrical coordinates, and marching in planes parallel to a coordinate plane. Ferri introduces the 3-D method of characteristics with the tetrahedral scheme and discards it as it involves a large amount of numerical work and makes the introduction of boundary conditions difficult. He turns to a reference-plane scheme similar in approach to that in his previous work.

Rakich (11) develops a reference-plane scheme in which the continuity and momentum equations are written in combined form in terms of orthogonal vectors with one lying in the streamwise direction. The compatibility equations for the bicharacteristic theory are first derived. Rakich notes two complications of three-dimensional method of characteristics — the presence of cross-derivative terms and the interpolation for data in the initial-value surface — which he feels make the bicharacteristic methods less desirable as the programming becomes cumbersome and the evaluation of cross-derivatives is generally of a lower accuracy than that of the basic calculations.

The preferred reference-plane method calculates the flowfield along the reference planes successively while marching in the axial direction. The compatibility equations from characteristic theory are solved along the projections of the bicharacteristics onto

the reference planes. In the numerical procedure, for ease of computation the mesh points lie along coordinate lines and on the shock and body surfaces (in this study), giving a non-orthogonal coordinate system.




In a reference plane an inverse method is used to calculate the flow at a mesh point from the interpolated data at the points of intersection of the characteristics through the mesh point with the data line. For the calculation of flow field conditions at each mesh point the intersection with the data line of the streamline projection is needed as well.

An Euler predictor-corrector iteration method is used to solve the difference equations in the reference plane. The reference planes are coupled by the cross-derivative terms appearing in the finite-differenced compatibility equations. An approximate value of the function carrying these cross-derivative terms on the initial-data line with derivatives evaluated there must first be assumed; then the cross-derivatives on the next data line can be calculated using the calculated values on all of the reference planes to then obtain the next approximation to the function with the cross-derivative terms needed to solve the compatibility equations. In practice Rakich found that such a global iteration was not necessary and that use of the derivatives evaluated on the initial-data line was sufficient.

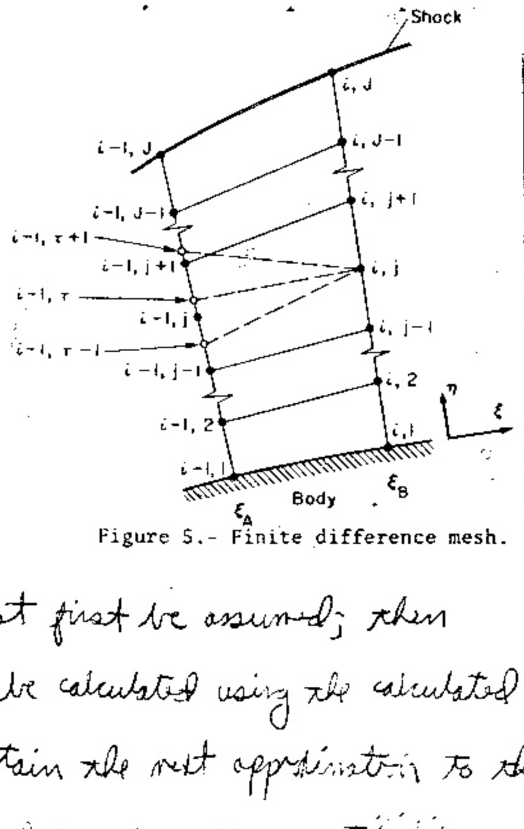



Figure 5.- Finite difference mesh.

Step size is determined by the CFL stability criterion for the linearized equations mentioned earlier, which states that the domain of dependence of a calculated point be included in the initial data. Rakich tested at a point where the maximum allowable step size was most likely to be smallest and took the actual step size to be 80% of that.

In his case of solving the flowfield about an inclined body of revolution the streamwise derivative was approximated by backward differencing (along the streamline), and radial derivatives were approximated by central differencing. Fourier-series approximations were applied to the flow variables in the circumferential direction, and circumferential derivatives were determined by directly differentiating the Fourier series. Computing time was approximately 6/10 that of Volyak and Hoffman based on an average time / point basis.

## CONCLUSIONS

The methods of Ferri (8,9) and Rakich (11) appear to offer the best means of approaching the problem of the three-dimensional method of characteristics for the potential equation. Ferri (8) basically treats the potential equation by a reference plane method, but his derivation in the text is extremely concise and perhaps can be more clearly expanded with the help of reference (9). Rakich in his work goes into great depth in covering differencing methods, application of boundary conditions, etc. and should be of use in treating the potential equation by a reference-plane method. Criticisms against Butler's bicharacteristic method would make one hesitate to choose that approach. The comments by Holt concerning the independence of bicharacteristic directions should be investigated should the Butler-Hoffman method be considered; nevertheless a study should be made of that part of Zucrow and Hoffman's <u>Gas Dynamics</u> Vol. II concerned with numerical application of three-dimensional method of characteristics for a current presentation (as the book becomes available).

Other finite difference schemes such as the explicit Lax-Wendroff or MacCormack, or the implicit Crank-Nicolson, ADI, or Approximate Factorization, are not studied. Such schemes are commonly used to solve Euler's equations or the Navier-Stokes equations. Perhaps the simplest approach would be to apply a second-order-accurate reference-plane scheme to the potential equation to find a solution for our supersonic shock-free flow domain.

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